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Quantum Phases and Connections

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Quantum phases and possible reductions of adiabatic connections are discussed. It is argued that only holonomy groups have physical meaning in that context.

Since the first recognition by Berry (1984) of the deep physical meaning of the wave functions' phases in the adiabatic approximation, the quantum phases have attracted the attention of both physicists and chemists (Zwanziger *et al.*, 1990). The analysis has been generalized to nonadiabatic and non-Abelian cases (Wilczek and Zee, 1984; Anandan and Aharonov, 1988; Ralston, 1989; Anandan, 1988; Anandan and Stodolski, 1987). These phases can be described in terms of connections and their holonomy groups (Simon, 1983; Page, 1987; Kobayashi and Nomizu, 1969). In this paper we would like to point out that, in some sense, every quantum evolution is geometrical.

Let us consider the Schrödinger equation (we have put $\hbar = 1$ for simplicity):

$$i\frac{\partial}{\partial t}\psi(t) = \mathscr{H}(t)\psi(t) \tag{1}$$

This equation can be solved in terms of the evolution operators $U(t, t_0)$ given by the equations

$$i\frac{\partial}{\partial t}U(t, t_0) = \mathscr{H}(t)U(t, t_0) \tag{2a}$$

$$-i\frac{\partial}{\partial t_0}U(t, t_0) = U(t, t_0)\mathscr{H}(t_0) \tag{2b}$$

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with the following initial condition:

$$U(t_0, t_0)=1 \tag{2c}$$

The solution to equation (2) has the form (Białynicki-Birula *et al.*, 1969)

$$\begin{aligned} U(t, t_0) &= T\text{-exp}\left[-i\int_{t_0}^{t} d\tau\, \mathcal{H}(\tau)\right] \\ &\equiv 1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{t} d\tau_n\, T(\mathcal{H}(\tau_1)\ldots\mathcal{H}(\tau_n)) \end{aligned} \tag{2d}$$

where T denotes the chronological ordering. It can be shown that the operators $U(t, t_0)$ are unitary and fulfill

$$U(t_2, t_1)U(t_1, t_0)=U(t_2, t_0) \tag{3}$$

Now, suppose that the quantum system depends on some (continuous) parameters a^i, $\mathcal{H}(t)\equiv\mathcal{H}(t, a^i)$. The Schrödinger equation can still be solved in terms of evolution operators, but now they depend on the parameters a^i. If we suppose that the Hamiltonian depends on time t only through the parameters $a^i=a^i(t)$, the solution to equations (2a) and (2b) will take the form

$$U(t, t_0)=P\exp\left[-i\int_{t_0}^{t} d\tau\, \mathcal{H}(a^i(\tau))\right] \tag{4}$$

where P denotes the path ordering. Let us notice that, in general, the condition $U(t_0, t_0)=1$ cannot be fulfilled. Suppose the evolution operators form a group G at each point a^i of the parameter space. This group is defined by "evolutions" along loops at a^i. In some cases [e.g., when G is a Lie group $\dim C\geq 2$, C being the parameter space (Kobayashi and Nomizu, 1969)] the group G can be realized as a holonomy group of some connection on the parameter space C. The question is how and to what extent is such a connection related to the Hamiltonian $\mathcal{H}$? To answer it, let us define the parallel transport operator (Dubrovin *et al.*, 1986) of the connection A:

$$S(\gamma, A)\equiv P\exp\left[\int_{t_0}^{t}\left(\frac{d}{d\tau}-\nabla_{\dot\gamma(\tau)}\right)d\tau\right] \tag{5}$$

where γ is a path and $\dot\gamma$ denotes the tangent vector field to γ, and $\nabla_{\dot\gamma}$ the covariant differential along $\dot\gamma$ of the connection A. The parallel transport is defined by the differential equation

$$\frac{d\psi^i(t)}{dt}+a^i_{j\mu}(t)\dot C^\mu(t)\psi^j(t) \tag{6}$$

where $a^i_{j\mu}$ denotes the coefficients (Kobayashi and Nomizu, 1969) of the connection A and $C(t) = (C^1(t), \ldots, C^n(t))$ is the appropriate path in the parameter space. Now,

$$\frac{d}{dt} - \nabla_{\dot{C}(t)} = -a^i_{j\mu}(t)\dot{C}^\mu(t) \tag{7a}$$

and

$$S(c, A) = P \exp\left[\int_{t_0}^{t} -a^i_{j\mu}(\tau)\dot{c}^\mu(\tau)\, d\tau\right] \tag{7b}$$

The crucial fact is that the element $S(c, A)$ of the structural group G does not depend on parametrization of the curve C. The evolution operator $U(t, t_0)$ given by formula (4) usually depends on the parametrization (the dynamic phase). This means that, in general, quantum evolution cannot be described in a geometrical way. When the assumptions of adiabatic evolution are fulfilled, the evolution of a state φ is governed by (Wilczek and Zee, 1984; Anandan and Aharonov, 1988)

$$\varphi(T) = \left\{\exp\left[-i\int_{t}^{T} w(t)\, dt\right]\right\} U(C)\varphi(0) \tag{8}$$

where $U(C)$ describes the geometrical phase. This geometrical phase is often expressed in terms of the adiabatic connection:

$$A_{ab,u} \doteq \left(\varphi_a \left| \frac{\partial}{\partial x_\mu} \varphi_b\right.\right) \tag{9}$$

as a parallel transport (Simon, 1893)

$$U(C) = P\text{-}\exp\left(-\int_c A_\mu \dot{c}^\mu\, dt\right) \tag{10}$$

We would like to stress that this connection has no apparent physical meaning: it may depend on the choice of representation. The presence of a $U(n)$ connection may also suggest that there is an underlying $U(n)$ gauge symmetry of the system. This is not so. Only the holonomy group is of physical importance. To show this, let us define the holonomy bundle of principal bundle $P(M, G)$ with a connection A (Kobayashi and Nomizu, 1969). Let $p \in P(M, G)$ and $H(p)$ denote the set of points that can be joined with p by a horizontal curve. It can be proved that (Kobayashi and Nomizu, 1969):

1. $H(p)$ is a reduced bundle with the holonomy group $h(p)$ as a structure group.
2. The connection A is reducible to a connection on $H(p)$.

This means that, unless the appropriate holonomy group is $U(n)$, the adiabatic holonomy manifold is "level."

Example. Suppose that the parameter manifold M is paracompact connected, but not necessarily simply connected. Let Γ denote a flat connection in $P(M, G)$. Let $p \in P(M, G)$ and $H(p)$ be the holonomy manifold through p. $H(p)$ is a principal fiber bundle with the holonomy group $h^{\Gamma}(p)$ as a structure group. It can be proved that:

1. $h^{\Gamma}(p)$ is discrete.
2. $h_0^{\Gamma}(p)$, the restricted holonomy group, is trivial.
3. Parallel transport defines a homomorphism of $\pi_1(M, \pi(p))$ onto $h^{\Gamma}(p)$.

This example shows that we may have nontrivial geometrical phases even when the adiabatic connection is flat (cf. the Bohm-Aharonov effect). This also proves that the formula (41) in Zwanziger *et al.* (1990) is correct only in the case of a simply connected parameter manifold. This analysis can obviously be generalized to quantum phases of a more general type.

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